Übungsstunde 5:

Themen:

- Basis in Vektorräumen
- Basen von linearen Abbildungen (Kern & Bild)
- Allgemeine Lösung eines LGS <=> Link zu DGL

$\underline{\underline{A}}\,\underline{x} = \underline{b}$

a) $\underline{\underline{A}}^{-1}$

b) $\underline{x} = \underline{\underline{A}}^{-1} \cdot \underline{b}$

Basis in Funktionsräumen:

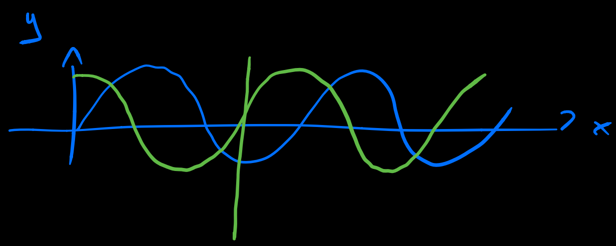
Frage: a) $\sin(x), \cos(x)$ ✓

$\triangleright$ $\sin(x) = a \cdot \cos(x) \quad a \in \mathbb{R}$

$\Leftrightarrow \quad a \cdot \sin(x) + b \cdot \cos(x) = 0$

$$x_1 \cdot \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + x_2 \cdot \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} = 0$$

triviale Lsg: $x_1 = x_2 = 0$



Beispiel: $\mathcal{P}_2$ (Polynome Grad ≤ 2)

$$\mathcal{B} = \{ b^{(1)} = 1,\ b^{(2)} = x,\ b^{(3)} = 3x^2 - 1 \} \quad , \quad \boxed{1, x, x^2}$$

$p(x) = \underline{2}x^2 + \underline{3}x \underline{- 4}$

$1 = b^{(1)}$

$x = b^{(2)}$

$x^2 = \frac{1}{3} b^{(3)} + \frac{1}{3} b^{(1)}$

$\quad = x^2 - \frac{1}{3} + \frac{1}{3}$

Kern & Bild linearer Abbildungen:

$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}$ eine $m \times n$ Matrix:

$\underline{\underline{A}}\,\underline{x} = x_1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + x_2 \cdot \begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} + \dots$

(i) $\underline{b} \in \mathrm{Im}(A) \Leftrightarrow \underline{\underline{A}}\,\underline{x} = \underline{b}$ lösbar ist. $b = \mathrm{span}\{a_1, a_2, \dots, a_n\}$

(ii) $\underline{x} \in \mathrm{Ker}(A) \Leftrightarrow \underline{\underline{A}}\,\underline{x} = 0$

(iii) $\mathrm{Ker}(A)$ UVR von $\mathbb{R}^n$

(iv) $\mathrm{Im}(A)$ UVR von $\mathbb{R}^m$

(v) $\dim(\mathrm{Ker}(A)) + \dim(\mathrm{Im}(A)) = n$

(vi) $\dim(\mathrm{Ker}(A)) = \dim(\mathrm{Im}(A^T))$

Beispiele: $F: \mathbb{R}^2 \to \mathbb{R}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1 - x_2$

$$\underline{\underline{A}} = \left.\overbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}^{2}\right\}1 \quad , \quad \underline{\underline{A}}\,\underline{x} = x_1 - x_2$$

$\mathrm{Ker}(\underline{\underline{A}})$: $\quad \underline{\underline{A}}\,\underline{x} = 0 \quad \Rightarrow \mathrm{Ker}(F) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 = x_2 \right\}$

$$= \left\{ t \cdot \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, t \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Basis des Kerns

$\mathrm{Im}(F) = \mathbb{R}$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -4 & 0 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$\mathrm{Ker}(A)$: $\quad \underline{\underline{A}}\,\underline{x} = 0$

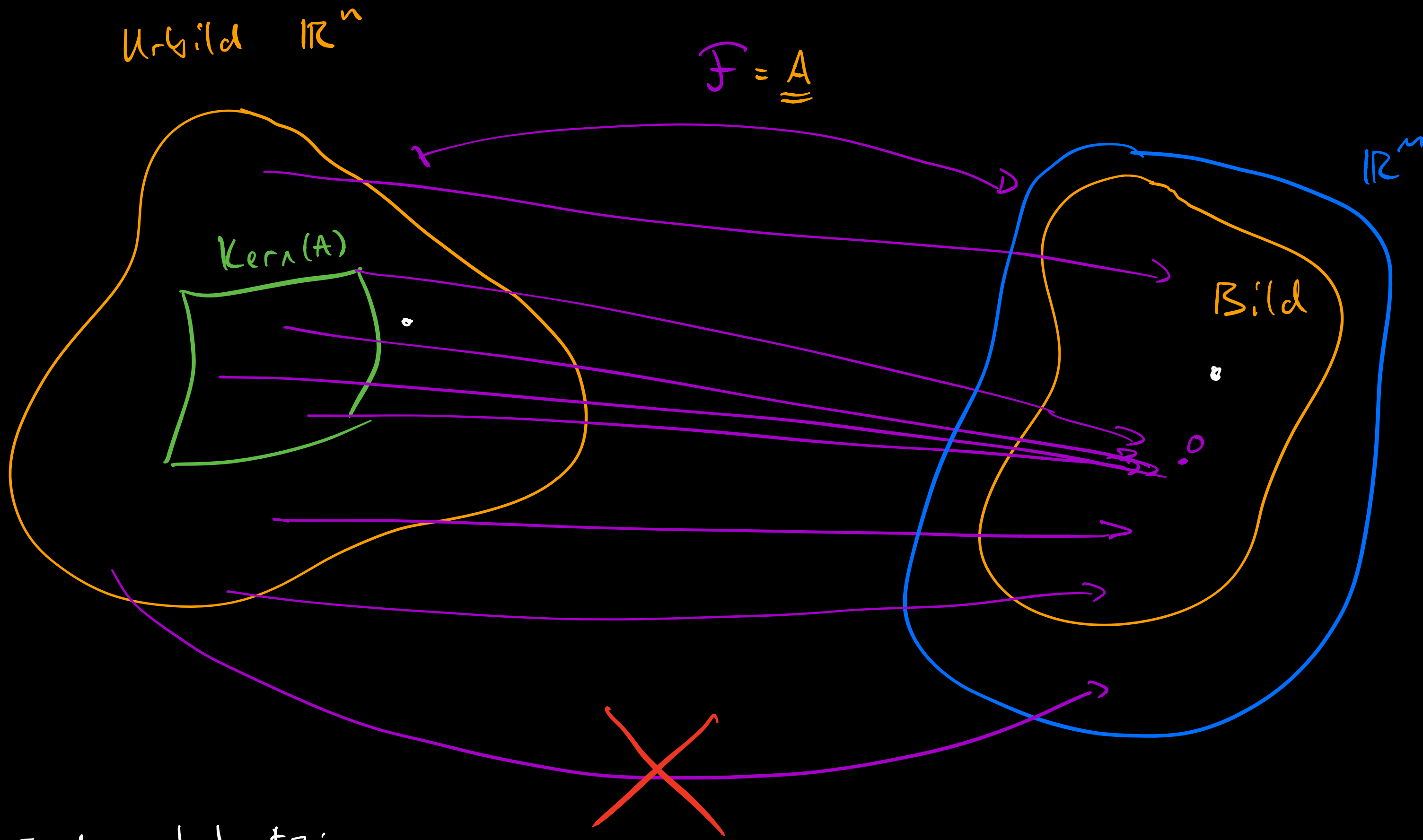
$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 0 \\ -4 & 0 & 1 & -3 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\text{III} - \text{I}]{\text{II} + 2\text{I}} \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_3 = s \in \mathbb{R} \\ x_4 = t \in \mathbb{R} \\ x_2 = \frac{3}{2}(t - s) \\ x_1 = \frac{1}{4}(s - 3t) \end{cases} \Rightarrow \mathrm{Ker}(A) = \left\{ s \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}, t, s \in \mathbb{R} \right\}$$

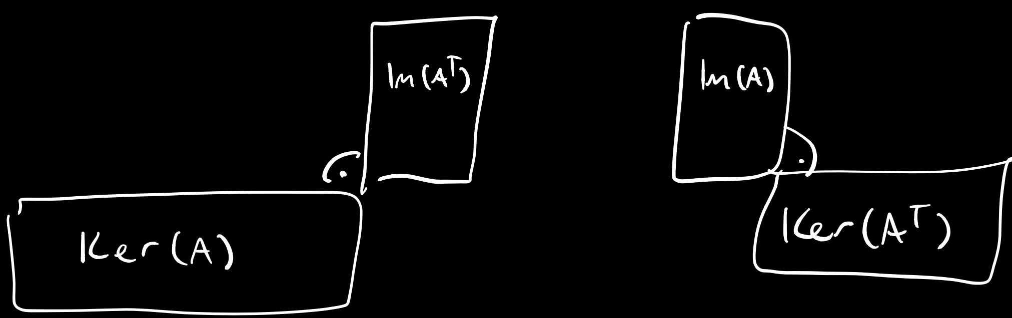
$$= \underline{\underline{\mathrm{span}\left\{ \begin{bmatrix} 1 \\ -6 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 0 \\ 4 \end{bmatrix} \right\}}}$$

$$\mathrm{Bild}(A) = \underline{\underline{\mathrm{span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}}$$

$$\underline{\underline{A}}^{m \times n}$$



Fundamentalsatz:



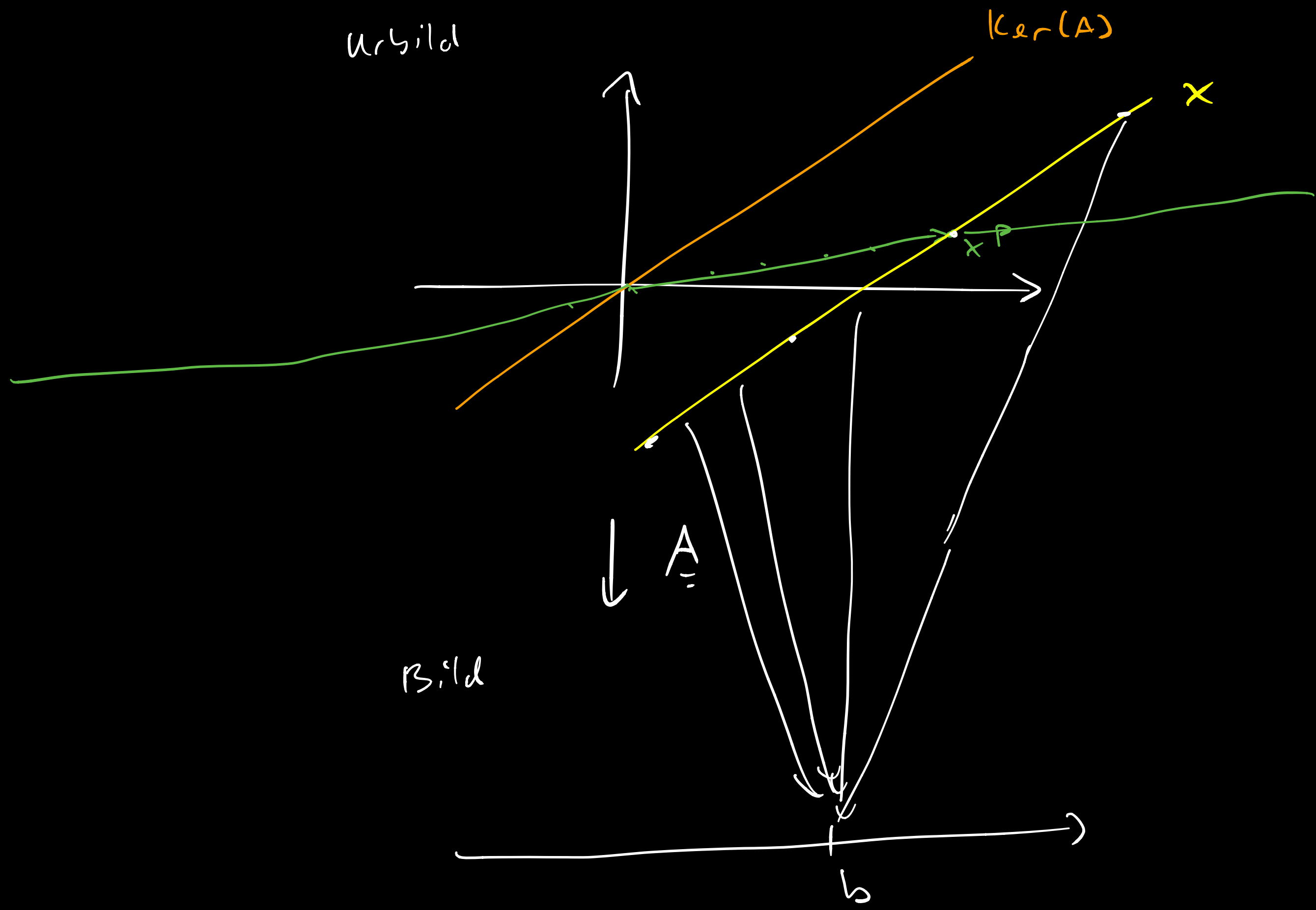
Allgemeine Lösung eines LGS / DGL

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \iff \underline{x}^p$$

$$\underline{x} = \underline{x}^p + \alpha\, \underline{x}^{h1} + \beta\, \underline{x}^{h2}$$

$$\mathrm{Ker}(A) = \mathrm{span}\{x^{h1}, x^{h2}\}$$

$$\Rightarrow \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(\underline{x}^p + \alpha \underline{x}^{h1} + \beta x^{h2}) = \underbrace{\underline{\underline{A}}\,\underline{x}^p}_{= b} + \alpha \underbrace{\underline{\underline{A}}\,\underline{x}^{h1}}_{0} + \beta \underbrace{\underline{\underline{A}}\,\underline{x}^{h2}}_{0}$$

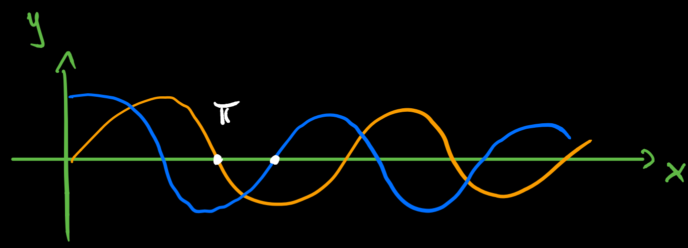


Urbild
Ker(A)
x
p
A
Bild
b

Übungsstunde 5:

Basis in Funktionsräumen:

Bsp: $\sin(x)$, $\cos(x)$



$$a \cdot \sin(x) + b \cdot \cos(x) = 0$$

$$a \cdot 0 + b \cdot \cos(\pi) = 0$$

$$a \in \mathbb{R}, \; b = 0$$

$$a \cdot \sin\left(\frac{3\pi}{2}\right) + 0 \cdot 0 = 0$$

$$a = 0$$

Bsp: $\mathcal{P}_2$: Polynome Grad ≤ 2, $B = \{ b^{(1)} = 1, \; b^{(2)} = x, \; b^{(3)} = 3x^2 - 1 \}$

$$p(x) = \underline{7}x^2 + \underline{5}x - \underline{3}$$

$$\underbrace{5(x^2 + x)} + 2x^2 - 3$$

TrivialBasis: Monome

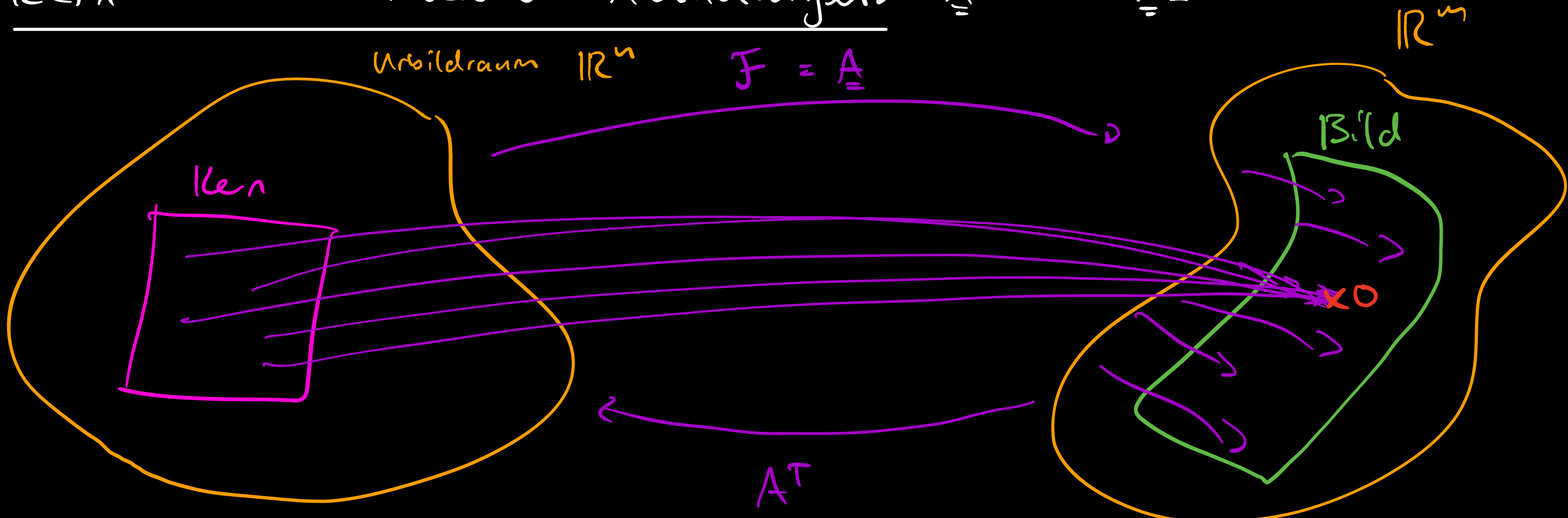
$1, x, x^2, x^3, \dots$

$$1 = b^{(1)} = 1$$

$$x = b^{(2)} = x$$

$$x^2 = \frac{1}{3} b^{(3)} + \frac{1}{3} b^{(1)} = \frac{1}{3}(3x^2 - 1) + \frac{1}{3} = x^2$$

Kern & Bild linearer Abbildungen: $\underline{A}^{m \times n}$ $\quad \underline{A}\underline{x} = \underline{b}$



$\underline{\underline{A}} = [\, a_1 \ a_2 \ \dots \ a_n \,]\}m$ $\qquad$ $\underline{\underline{A}}\,\underline{x} = x_1 [a_1] + x_2 [a_2] + \dots$

i) $b \in Im(A) \Leftrightarrow \underline{\underline{A}}\,\underline{x} = \underline{b}$ ist ein lösbares LGS.

$b = span\{a_1, a_2, \dots, a_n\}$

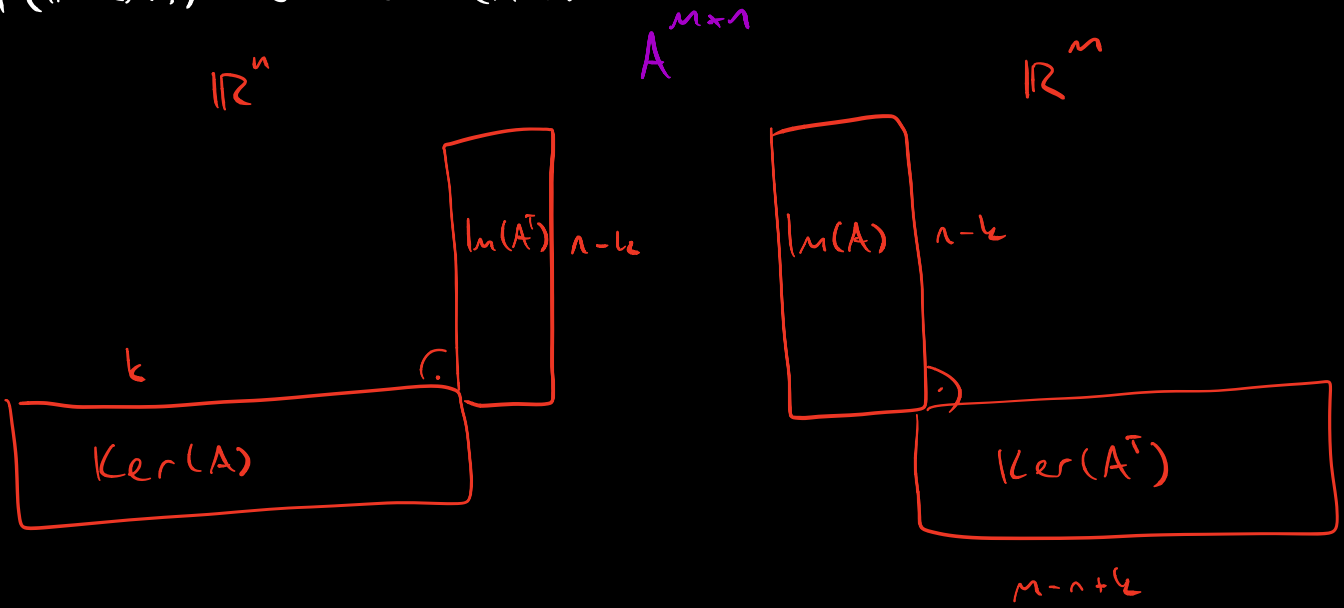
ii) $x \in Ker(A) \Leftrightarrow x$ eine Lösung zum HLGS $\underline{\underline{A}}\,\underline{x} = 0$

iii) $Ker(A)$ UVR $\mathbb{R}^n$

iv) $Im(A)$ UVR $\mathbb{R}^m$

v) $\dim(Ker(A)) + \dim(Im(A)) = n$

vi) $\dim(Im(A)) = \dim(Im(A^T))$



$\underline{\underline{A}}\,\underline{x} = \underline{b}$

$\Updownarrow$

$\underline{x} \perp (\underline{\underline{A}}^T \underline{y} = 0)$

Bild $\Leftrightarrow$ Spaltenraum

Bild (A^T) $\Leftrightarrow$ Zeilenraum

<u>Berechnung:</u>

Bsp: $F: \mathbb{R}^2 \to \mathbb{R}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1 - x_2$

$\underline{\underline{A}} = [1 \ \ -1]$

Kern: $\underline{\underline{A}}\,\underline{x} = 0$: $\boxed{1} \ -1 \mid 0$

$x_2 = t$

$x_1 = t$ $\Rightarrow Ker(A) = \left\{ t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$

$= span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Bild: $\operatorname{Im}(A) = \underline{\underline{\mathbb{R}}}$

Bsp:

$$\underline{A} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -4 & 0 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\operatorname{Ker}(A): \left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 0 \\ -4 & 0 & 1 & -3 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[\text{III}-\text{I}]{\text{II}+2\text{I}} \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \boxed{2} & 1 & 1 & 0 & 0 \\ 0 & \boxed{2} & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$x_3 = t$

$x_4 = s$

$x_2 = \frac{1}{2}(3s - 3t)$

$2x_1 + \frac{3}{2}s - \frac{3}{2}t + t = 0 \iff x_1 = \frac{1}{4}(t - 3s)$

$$\Rightarrow \begin{bmatrix} \frac{1}{4}t - \frac{3}{4}s \\ \frac{3}{2}s - \frac{3}{2}t \\ 0s\ \ 1t \\ 0t\ \ 1s \end{bmatrix}$$

$$\Rightarrow \operatorname{Ker}(A) = \left\{ t \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix},\ t, s \in \mathbb{R} \right\}$$

$$= \underline{\underline{\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -6 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 0 \\ 4 \end{bmatrix} \right\}}}$$

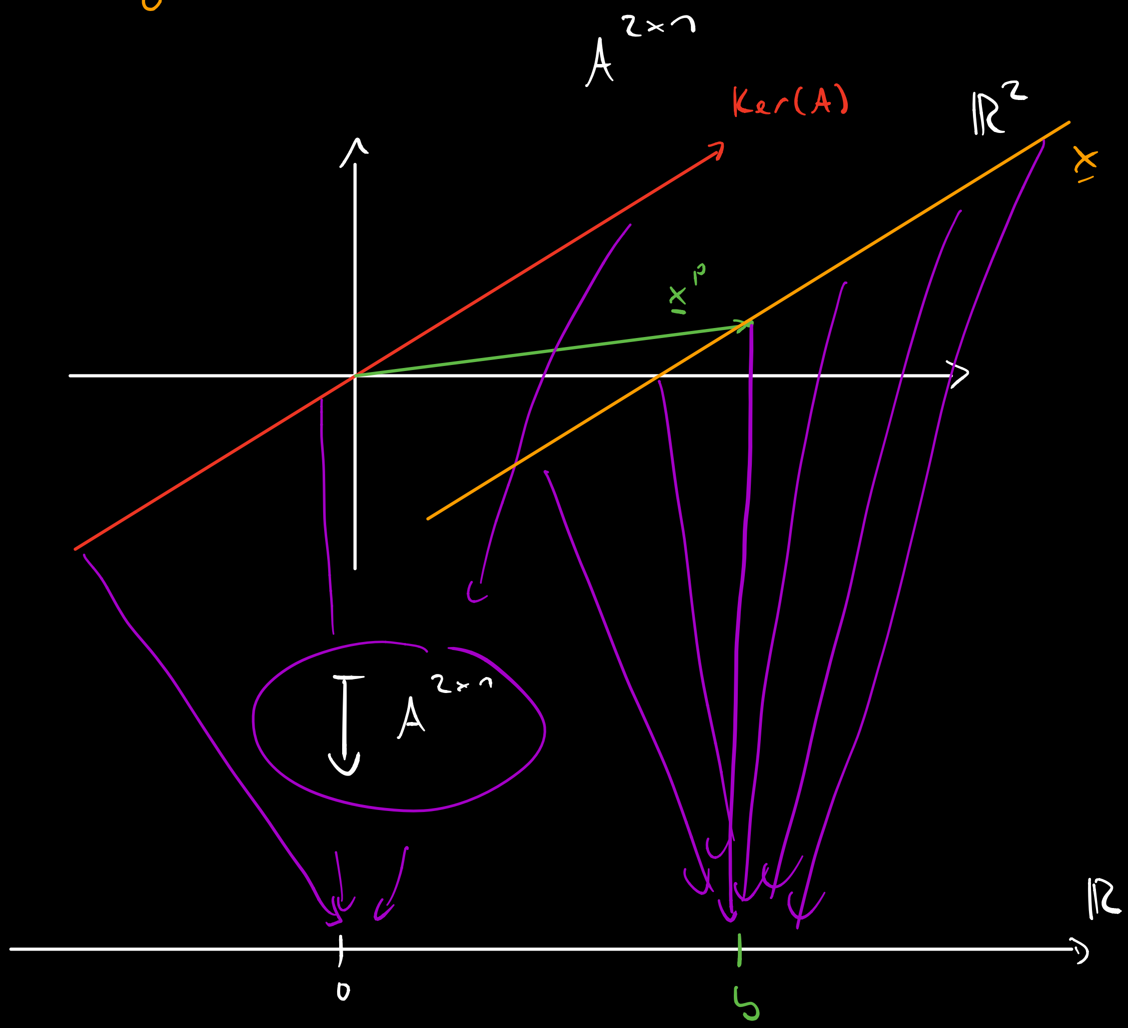
Bild:

$$\underline{\underline{\operatorname{Im}(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}}$$

Allgemeine Lösungen (LGS & DGL):

$\underline{\underline{A}}\,\underline{x} = \underline{b}$, $\quad \underline{x} = \underline{x}^{p} + \alpha\,\underline{x}^{h1} + \beta\,\underline{x}^{h2}$

$$\begin{aligned}
\underline{\underline{A}}\,\underline{x} &= \underline{\underline{A}}\left(\underline{x}^{p} + \alpha\,\underline{x}^{h1} + \beta\,\underline{x}^{h2}\right) \\
&= \underbrace{\underline{\underline{A}}\,\underline{x}^{p}}_{b} + \underbrace{\alpha\,\underline{\underline{A}}\,\underline{x}^{h1}}_{0} + \underbrace{\beta\,\underline{\underline{A}}\,\underline{x}^{h2}}_{0} \\
&= b
\end{aligned}$$



$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

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korrigieren